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HW1

PHYS 515B
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1. Jackson 11.3:
   Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

Solution. WLOG, construct transformations in x-direction with velocities v_1 and v_2. Define

$$\begin{aligned} \beta_i &= \frac{v_i}{c} \\ \gamma_i &= \frac{1}{\sqrt{1-\beta_i^2}} \qquad i = 1, 2 \end{aligned}$$

The transformations then become

$$A_1 = \begin{bmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; A_2 = \begin{bmatrix} \gamma_2 & -\beta_2\gamma_2 & 0 & 0 \\ -\beta_2\gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Taking the matrix product $A_2 A_1$, the net Lorentz transformation is

$$A_{net} = \begin{bmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & -\gamma_1\gamma_2(\beta_1+\beta_2) & 0 & 0 \\ -\gamma_1\gamma_2(\beta_1+\beta_2) & \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma_{net} & -\beta_{net}\gamma_{net} & 0 & 0 \\ -\beta_{net}\gamma_{net} & \gamma_{net} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Working out the (messy) algebra yields

$$\begin{aligned} \gamma_{net} &= \gamma_1\gamma_2(1+\beta_1\beta_2) \\ &= \frac{1}{\sqrt{1-\left(\frac{v_1+v_2}{c(1+\frac{v_1v_2}{c^2})}\right)^2}} = \frac{1}{\sqrt{1-\beta_{net}^2}} \\ \beta_{net} &= \frac{v_1+v_2}{c(1+\frac{v_1v_2}{c^2})} = \frac{v_{net}}{c} \\ (\implies)\; v_{net} &= \frac{v_1+v_2}{(1+\frac{v_1v_2}{c^2})} \quad \square \end{aligned}$$

2. Jackson 11.9 a and b:
   An infinitesimal Lorentz transformation and its inverse can be written as

$$\begin{aligned} x'^{\alpha} &= (g^{\alpha\beta} + \epsilon^{\alpha\beta})x_\beta \qquad (1) \\ x^{\alpha} &= (g^{\alpha\beta} + \epsilon'^{\alpha\beta})x'_\beta \qquad (2) \end{aligned}$$

where $\epsilon^{\alpha\beta}$ and $\epsilon'^{\alpha\beta}$ are infinitesimal.

a.) Show from the definition of the inverse that $\epsilon'^{\alpha\beta} = -\epsilon^{\alpha\beta}$.

Solution. Start with eqn. (1) and take Lorentz transformation

$$\begin{aligned} x''^{\alpha} &= (g^{\alpha\beta} + \epsilon'^{\alpha\beta})x'_{\beta} \\ &= (g^{\alpha\beta} + \epsilon'^{\alpha\beta})g_{\beta\gamma}x'^{\gamma} \\ &= (g^{\alpha\beta} + \epsilon'^{\alpha\beta})g_{\beta\gamma}(g^{\gamma\delta} + \epsilon^{\gamma\delta})x_{\delta} \end{aligned}$$

We can now manipulate the last line above to get the equality

$$x''^{\alpha} = (g^{\alpha\beta} + \epsilon'^{\alpha\beta})g_{\beta\gamma}(g^{\gamma\delta} + \epsilon^{\gamma\delta})g_{\delta\eta}x^{\eta}. \tag{3}$$

Now use the definition of the inverse

$$x''^{\alpha} = \delta^{\alpha}{}_{\eta}x^{\eta}. \tag{4}$$

Compare (3) to (4):

$$\begin{aligned} \delta^{\alpha}{}_{\eta} &= (g^{\alpha\beta} + \epsilon'^{\alpha\beta})g_{\beta\gamma}(g^{\gamma\delta} + \epsilon^{\gamma\delta})g_{\delta\eta} \\ &= [\delta^{\alpha}{}_{\gamma}(g^{\gamma\delta} + \epsilon^{\gamma\delta}) + \epsilon'^{\alpha\beta}(g^{\gamma\delta} + \epsilon^{\gamma\delta})]g_{\delta\eta} \\ &= [g^{\alpha\delta} + \epsilon^{\alpha\delta} + \epsilon'^{\alpha\delta}(\delta^{\delta}{}_{\beta} + g_{\beta\gamma}\epsilon^{\gamma\delta})]g_{\delta\eta} \\ &= (g^{\alpha\delta} + \epsilon^{\alpha\delta} + \epsilon'^{\alpha\delta} + \epsilon'^{\alpha\beta}g_{\beta\delta}\epsilon^{\gamma\delta})g_{\delta\eta} \end{aligned}$$

The last step leads to a slight difficulty. However, due the infinitesimal size of elements of the ϵ-tensors we are justified in dropping terms that are quadratic or higher, i.e. $\epsilon'^{\alpha\beta}g_{\beta\delta}\epsilon^{\gamma\delta} \to 0$. Now, the final step is:

$$\delta^{\alpha}{}_{\eta} = \delta^{\alpha}{}_{\eta} + g_{\delta\eta}(\epsilon^{\alpha\delta} + \epsilon'^{\alpha\delta})$$

Change index δ to β

$$= \delta^{\alpha}{}_{\eta} + g_{\beta\eta}(\epsilon^{\alpha\beta} + \epsilon'^{\alpha\beta})$$

For the equality to hold, $\epsilon'^{\alpha\beta} = -\epsilon^{\alpha\beta} \quad \square$.

b.) Show from the preservation of the norm that $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$.

Solution. Start with the norm condition for Lorentz transformations and then do tensor operations using relations given from (1) and (2) :

$$\begin{aligned} x^{\alpha}x_{\alpha} &= x'^{\alpha}x'_{\alpha} \\ &= (g^{\alpha\beta} + \epsilon^{\alpha\beta})x_{\beta}x'_{\alpha} \\ &= (g^{\alpha\beta} + \epsilon^{\alpha\beta})x_{\beta}g_{\alpha\gamma}x'^{\gamma} \\ &= (g^{\alpha\beta} + \epsilon^{\alpha\beta})x_{\beta}g_{\alpha\gamma}(g^{\gamma\delta} + \epsilon^{\gamma\delta})x_{\delta} \\ &= (g^{\alpha\beta} + \epsilon^{\alpha\beta})[g_{\alpha\gamma}(g^{\gamma\delta} + \epsilon^{\gamma\delta})]x_{\beta}g_{\delta\eta}x^{\eta} \\ &= (g^{\alpha\beta} + \epsilon^{\alpha\beta})[\delta^{\delta}{}_{\alpha} + g_{\alpha\gamma}\epsilon^{\gamma\delta}]g_{\delta\eta}x_{\beta}x^{\eta} \\ &= (g^{\alpha\beta} + \epsilon^{\alpha\beta})[g_{\delta\eta} + g_{\alpha\delta}\epsilon^{\gamma\delta}g_{\delta\eta}]x_{\beta}x^{\eta} \\ &= (\delta^{\beta}{}_{\eta} + \delta^{\beta}{}_{\gamma}\epsilon^{\gamma\delta}g_{\delta\eta} + \epsilon^{\alpha\beta}g_{\alpha\eta})x_{\beta}x^{\eta} \\ &= x_{\beta}x^{\beta} + (\epsilon^{\beta\delta}g_{\delta\eta} + \epsilon^{\alpha\beta}g_{\alpha\eta})x_{\beta}x^{\eta} \end{aligned}$$

Change indices α to β and δ to α

$$
\begin{aligned}
&= x_\alpha x^\alpha + (\epsilon^{\beta\alpha} g_{\alpha\eta} + \epsilon^{\alpha\beta} g_{\alpha\eta}) x_\beta x^\eta \\
&= x_\alpha x^\alpha + (\epsilon^{\beta\alpha} + \epsilon^{\alpha\beta}) g_{\alpha\eta} x_\beta x^\eta \\
&= x_\alpha x^\alpha + (\epsilon^{\beta\alpha} + \epsilon^{\alpha\beta}) x_\alpha x^\beta
\end{aligned}
$$

In order for the equality to hold, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ $\square$.

3. Find the matrix for the Lorentz transformation consisting of a boost v_x in the x-direction followed by a boost v_y in the y-direction. Show that the boost performed in the revise order (first v_y and then v_x) would give a different transformation.

   Solution. Define the quantities

$$
\begin{aligned}
\beta_1 &= \frac{v_x}{c} \\
\beta_2 &= \frac{v_y}{c} \\
\gamma_1 &= \frac{1}{\sqrt{1-\beta_1^2}} \\
\gamma_2 &= \frac{1}{\sqrt{1-\beta_2^2}}
\end{aligned}
$$

   Define the Lorentz transformation matrices

$$
A_1 = \begin{bmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & \gamma_2 & 0 & 0 \\ -\beta_2\gamma_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

   Performing v_x boost first yields the transformation

$$
A_{21} = A_2 A_1 = \begin{bmatrix} \gamma_1\gamma_2 & -\beta_1\gamma_1\gamma_2 & -\beta_2\gamma_2 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ -\beta_1\gamma_1\gamma_2 & -\beta_1\beta_2\gamma_1\gamma_2 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

   Performing v_y boost first yields the transformation

$$
A_{12} = A_1 A_2 = \begin{bmatrix} \gamma_1\gamma_2 & -\beta_1\gamma_1 & -\beta_2\gamma_1\gamma_2 & 0 \\ -\beta_1\gamma_1\gamma_2 & \gamma_1 & -\beta_1\beta_2\gamma_1\gamma_2 & 0 \\ -\beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

   Clearly, the two transformations are not identical. However, $A_{12} = (A_{21})^T$.

4. The Lorentz tranformation from frame O to O' is

$$x'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta}$$

where $A^{\alpha}{}_{\beta}$ has components

$$\begin{aligned}
A^0{}_0 &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\
A^0{}_j &= A^j{}_0 = -\frac{\frac{v_j}{c}}{\sqrt{1-\frac{v^2}{c^2}}} \\
A^j{}_k &= A^k{}_j = \delta_{jk} + \frac{v_j v_k}{v^2}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)
\end{aligned}$$

where v is the velocity of O' with respect to O.

For covariant vector, the transformation is

$$x'_{\alpha} = A_{\alpha}{}^{\beta} x_{\beta}$$

Find $A_{\alpha}{}^{\beta}$.

Solution.

$$\begin{aligned}
x'_{\alpha} &= g_{\alpha\beta} x'^{\beta} = g_{\alpha\beta} A^{\beta}{}_{\delta} \\
(\implies) &\quad A_{\alpha}{}^{\beta} x_{\beta} = g_{\alpha\beta} A^{\beta}{}_{\gamma} x^{\gamma} \\
(\implies) &\quad A_{\alpha}{}^{\beta} g_{\beta\gamma} x^{\gamma} = g_{\alpha\beta} A^{\beta}{}_{\gamma} x^{\gamma} \\
(\implies) &\quad A_{\alpha}{}^{\beta} g_{\beta\gamma} g^{\beta\gamma} = g_{\alpha\beta} A^{\beta}{}_{\gamma} g^{\beta\gamma} \\
(\implies) &\quad A_{\alpha}{}^{\beta} = g_{\alpha\beta} A^{\beta}{}_{\gamma} g^{\beta\gamma}
\end{aligned}$$

Using the definition for $g_{\alpha\beta}$

$$g_{\alpha\beta} = g^{\beta\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Performing the multiplications in (46) shows that all matrix entries will be the same except for

$$A_0{}^j = A_j{}^0 = -A^0{}_j = -A^j{}_0 = \frac{\frac{v_j}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The total transformation can be described by

$$
\begin{aligned}
A_0{}^0 &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\
A_0{}^j &= A_j{}^0 = \frac{\frac{v_j}{c}}{\sqrt{1-\frac{v^2}{c^2}}} \\
A_j{}^k &= A_k{}^j = \delta_{jk} + \frac{v_j v_k}{v^2}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)
\end{aligned}
$$